Tutorial 1. Vector space, subspace, span and linear independence.

Def 1.1. (field).

A <u>field</u> is a set $\mathbb{F}$ with two binary operations $+, \cdot : \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ satisfying the following axioms: ($a,b,c\in\mathbb{F}$ unless speified).

(F1). $(a+b)+c = a+(b+c)$. associativity for $+$

$(ab)c = a(bc)$. associativity for $\cdot$.

(F2). $a+b = b+a$ commutativity for $+$

$ab = ba$ commutativity for $\cdot$

(F3). $\exists\ x\in\mathbb{F}$ s.t. $x+a=a$ for all $a\in\mathbb{F}$. Denote x as "0". identity for $+$

$\exists\ y\in\mathbb{F}$ s.t. $y\cdot a = a$ for all $a\in\mathbb{F}$ Denote y as "1". identity for $\cdot$.

($\mathbb{F}$4). $\forall a\in\mathbb{F}$, $\exists b\in\mathbb{F}$ s.t. $a+b=0$. Denote b as "$-a$". inverse for $+$.

$\forall a\neq 0$ in $\mathbb{F}$, $\exists b\in\mathbb{F}$ s.t. $ab=1$ Denote b as a^{-1}. inverse for $\cdot$

($\mathbb{F}$5). $a(b+c) = ab+ac$ distributivity of $\cdot$ over $+$.

Remark. (i) Note that ($\mathbb{F}$1) – ($\mathbb{F}$3) is symmetric for $(\mathbb{F},+)$ and $(\mathbb{F},\cdot)$.

($\mathbb{F}$4) is not as we require $a\neq 0$.

($\mathbb{F}$5) is not either.

(ii). We usually say the triple $(\mathbb{F},+,\cdot)$ is a field as there can be more than one definition of $+$ and $\cdot$ making the same set into a different field.

If there is no danger of ambiguity, we will only write $\mathbb{F}$.

(iii). If you take MATH 2070, $(\mathbb{F},+)$ and $(\mathbb{F}\setminus\{0\},\cdot)$ are abelian groups.

Def 1.2. (vector space).

Let $(\mathbb{F},\oplus,\cdot)$ be an arbitrary field. A <u>vector space</u> is a set V with a binary operation $+: \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ and a <u>scalar multiplication</u> $*: \mathbb{F}\times V\to V$ satisfying the following axioms: ($u,v,w\in V$, $a,b\in\mathbb{F}$ unless specified).

(VS1). $u+(v+w) = (u+v)+w$. associativity for $+$.

(VS2). $u+v = v+u$ commutativity for $+$.

(VS3) $\exists x\in V$ s.t. $x+u=u$ for all $u\in V$ Denote x as "0" identity for $+$

(VS4) $\forall u\in V$, $\exists y\in V$ s.t. $u+y=0$ Denote y as "$-u$" inverse for $+$.

((VS1) – (VS4) make $(V,+)$ an abelian group. Note that $*$ is not relevant so far.)

(VS 5) $a*(b*v) = (a \underset{\text{in } \mathbb{F}}{\cdot} b)*v$ compatibility of $\cdot$ in $\mathbb{F}$ and $*$.

(VS 6) $1*v = v$ compatibility of $1 = 1_{\mathbb{F}}$ for $\cdot$ in $\mathbb{F}$ and $*$.

(VS 7) $a*(u+v) = a*u + a*v$ distributivity of $*$ over $+$ in V.

(VS 8) $(a \oplus b)*v = a*v + b*v$ distributivity of $*$ over $\oplus$ in $\mathbb{F}$.

Remark. (i) It is very important to distinguish $\oplus$ in $\mathbb{F}$ with $+$ in V ! and $\cdot$ in $\mathbb{F}$ with $* : \mathbb{F} \times V \to V$!!!

(ii) If you take MATH 2070, $(V,+)$ is an abelian group.

(iii) Once you are fluent, we write $\oplus$ as $+$, $*$ as $\cdot$, keeping in mind that they are different. We shall do it from next tutorial.

Q1. Show the following properties for a vector space V over $\mathbb{F}$.

(i) $0 \in V$ is unique, called the zero vector.

(ii) $\forall u \in V$, the inverse $-u$ is unique, called the additive inverse of u.

(iii) If $u+v = w+v$, then $u = w$. cancellation law.

(iv). $0*v = 0$ for all $v \in V$, where the first $0 = 0_{\mathbb{F}}$ in $\mathbb{F}$, and second $0 = 0_V$ in V.

(v). $a \cdot 0 = 0$ for all $a \in \mathbb{F}$. where both $0 = 0_V$.

(vi). $(-1)*v = -v$ for all $v \in V$, where -1 is the additive inverse to 1 with respect to $\oplus$ in $\mathbb{F}$ and $-v$ is the additive inverse to v with respect to $+$ in V.

Pf. Exercise. We will just show (iv).

$$0_V + 0_{\mathbb{F}}*v = 0_{\mathbb{F}}*v = (0_{\mathbb{F}} + 0_{\mathbb{F}})*v = 0_{\mathbb{F}}*v + 0_{\mathbb{F}}*v$$

From (iii), we have $0_V = 0_{\mathbb{F}}*v$.

Q2. Further properties.

(vii). $-(-v) = v$.

(viii). $(-a)*v = -(a*v) = a*(-v)$. $a \in \mathbb{F}$, $v \in V$.

(ix). If $a*v = 0$, then either $a = 0$ or $v = 0$

(x). For $u, v \in V$, $\exists!\ w \in V$ s.t. $u+w = v$.

(xi). $(a \oplus b)*(u+v) = a*u + a*v + b*u + b*v$, $a, b \in \mathbb{F}$, $u, v \in V$.

Q3. Let X be an arbitrary set, with power set $\mathcal{P}(X)$. Then the boolean algebra $(\mathcal{P}(X), \Delta)$ is an $\mathbb{F}_2$- vector space.

We will explain every terminology here.

(i) $\mathcal{P}(X)$ is a <u>boolean algebra</u> because it is closed under finite union $\cup$, finite intersection $\cap$ and complement $\setminus$, i.e.,

For $A, B \in \mathcal{P}(X)$, we have

- $A \cup B \in \mathcal{P}(X)$.
- $A \cap B \in \mathcal{P}(X)$
- $X \setminus A \in \mathcal{P}(X)$.

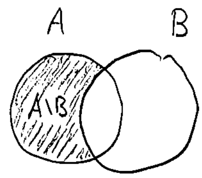
(ii). $\mathbb{F}_2$ is a <u>field</u>.

- As set, $\mathbb{F}_2 = \{0, 1\}$.
- addition table and multiplication table.

$\oplus$	0	1
0	0	1
1	1	0

$\bullet$	0	1
0	0	0
1	0	1

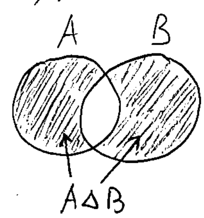
One may verify this is indeed a field.



(iii). Define the <u>symmetric difference</u> $\Delta : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$

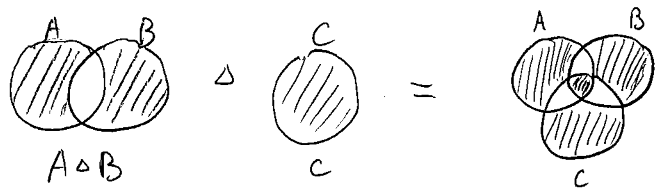
$$(A, B) \longmapsto (A \setminus B) \cup (B \setminus A).$$

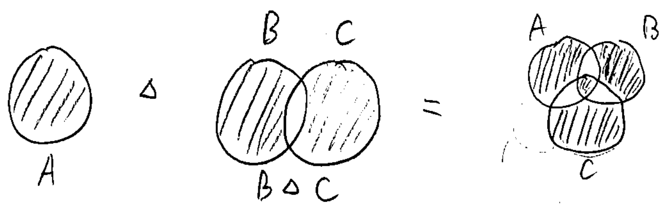
where $A \setminus B$ is defined to be $A \cap (X \setminus B)$ using (i).

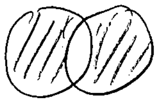


(iv). Check $(\mathcal{P}(X), \Delta)$ is a vector space over $(\mathbb{F}_2, \oplus, \cdot)$.

(VS1) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

LHS: 

RHS: 

(VS2). $A \triangle B = B \triangle A$.  obvious.

(VS3). $\exists \phi \in P(X)$, $\phi \triangle A = A$. as $(\phi \setminus A) \cup (A \setminus \phi) = \phi \cup A = A$ identity is ϕ.

(VS4). $\forall A \in P(X)$, $\exists A \in P(X)$ s.t. $A \triangle A = \phi$. as $(A \setminus A) \cup (A \setminus A) = \phi \cup \phi = \phi$. inverse is itself.

Define $* : \mathbb{F}_2 \times P(X) \to P(X)$

$$(0, A) \mapsto \phi.$$
$$(1, A) \mapsto A.$$

(VS5). $a * (b * A) = (a \cdot b) * A$.

a	b	LHS	LHS.
0	0	$0*(0*A) = 0*\phi = \phi$	$(0\cdot 0)*A = 0*A = \phi$.
0	1	$0*(1*A) = 0*A = \phi$	$(0\cdot 1)*A = 0*A = \phi$.
1	0	$1*(0*A) = 1*\phi = \phi$	$(1\cdot 0)*A = 0*A = \phi$.
1	1	$1*(1*A) = 1*A = A$.	$(1\cdot 1)*A = 1*A = A$.

(VS6). $1 * A = A$ directly from definition.

(VS7). $a * (A \triangle B) = (a * A) \triangle (a * B)$

$a=0$. $LHS = \phi$. $RHS = \phi \triangle \phi = \phi$.

$a=1$. $LHS = A \triangle B$. $RHS = A \triangle B$.

(VS8). $(a \oplus b) * A = (a * A) \triangle (b * A)$

a	b	LHS	RHS
0	0	$(0\oplus 0)*A = \phi$	$(0*A)\triangle(0*A) = \phi \triangle \phi = \phi$
0	1	$(0\oplus 1)*A = 1*A = A$.	$(0*A)\triangle(1*A) = \phi \triangle A = A$.
1	0	$(1\oplus 0)*A = (0\oplus 1)*A = A$.	$(1*A)\triangle(0*A) = (0*A)\triangle(1*A) = A$. ← alternative method using commutivity of $\oplus$ and $\triangle$.
1	1	$(1\oplus 1)*A = 0*A = \phi$.	$(1*A)\triangle(1*A) = A \triangle A = \phi$

Q4. Recall that $(\mathbb{R},+)$ is a vector space over itself $(\mathbb{R},+,\cdot)$, where $* = \cdot$.
Show that it is not possible to extend it as a vector space over $(\mathbb{C},+,\cdot)$.

pf. [(VS1)-(VS4) do not involve scalar multiplication, so it should be fine.
(VS6) is also fine as $1_{\mathbb{R}} = 1_{\mathbb{C}} = 1$.]

Suppose for $z \in \mathbb{C}$, $x \in \mathbb{R}$, we can define $* : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ s.t. $z * x \in \mathbb{R}$ and when $z \in \mathbb{R}$, $z * x = z \cdot x$. Then

$$f : \mathbb{C} \to \mathbb{R}$$
$$x \mapsto x * 1$$

is an injection, as $x*1 = y*1$ implies $(x-y)*1 = 0$ so $x=y$. It is also surjective as $f|_{\mathbb{R}} = id_{\mathbb{R}}$.
But this is not possible because $f(i) = f(x)$ for some $x \in \mathbb{R}$ implies $i = x \in \mathbb{R}$ ($\Rightarrow\Leftarrow$). $\square$

Subspaces.

Def. 1.3. A vector <u>subspace</u> of a vector space $(V,+)$ over $\mathbb{F}$ is a subset $U \subseteq V$ that is also a vector space where $+$ and scalar multiple $*$ is restricted to U.

$$+|_U : U \times U \to U. \qquad *|_U : \mathbb{F} \times U \to U.$$

Remark. It will be clumsy to check (VS1)-(VS8) every time, as most of them are automatic.

First, we need $+|_U$ and $*|_U$ is well-defined, i.e.,

☆ (i). For $u, v \in U$, $u+v \in U$.

☆ (ii). For $a \in \mathbb{F}$, $u \in U$, $a*u \in U$.

Next, we need (VS1)-(VS8).

(VS1) automatic.

(VS2) automatic.

(VS3). Not automatic, so we need

☆ (iii). $\exists\, 0 \in U$ s.t. $0+u=u$ for all $u \in U$.

(VS4) follow from (ii) and $-u = (-1)*u$

(VS5)-(VS8) automatic.

Hence we only need to check (i)-(iii) in practice.

Span

(From now on we always assume V is over a field $\mathbb{F}$ without mentioning the field.)

Def. 1.4. (linear combination)

Let V be a vector space, and $v_i \in V$ for $i \in I$ ($|I|$ may be infinite). An <u>$\mathbb{F}$-linear combination</u> of $\{v_i\}_{i\in I}$ is

$$v := \sum_{i \in I} a_i * v_i \quad \text{for some } a_i \in \mathbb{F} \text{ and only } \underline{\text{finitely}} \text{ many } a_i \neq 0$$

As the summation is finite, we have $v \in V$.

Def. 1.5. (span).

Let V be a vector space, and $S \subseteq V$ be a subset. We define the <u>span</u> of S, denoted $\langle S \rangle$ or span S to be all $\mathbb{F}$-linear combinations in S.

$$\operatorname{span} S := \left\{ \sum_{i \in I} a_i * s_i \;\middle|\; s_i \in S, \text{ only finitely many } a_i \neq 0,\ a_i \in \mathbb{F} \right\}$$

Linear independence

Def 1.6. (linear independence).

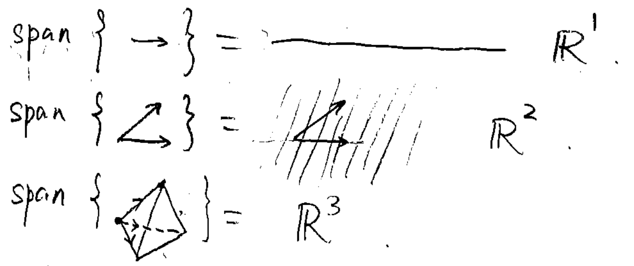
Let V be a vector space, and $S \subseteq V$. We say S is a <u>linearly independent</u> subset if for every $v_1, \cdots, v_n \in V$

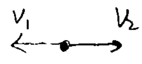
$$\sum_{i=1}^{n} a_i * v_i = 0 \text{ implies } a_1 = a_2 = \cdots = a_n = 0, \quad a_i \in \mathbb{F}.$$

i.e., every finite subset of S is linearly independent. We say S <u>linearly dependent</u> otherwise.

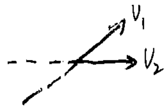
Remark. Geometric interpretation.

Let $V = \mathbb{R}^n$, $\mathbb{F} = \mathbb{R}$.

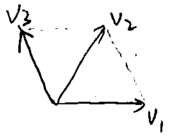




are linearly dependent, as $v_1 + (-1)v_2 = 0$, but $(1, -1) \neq 0$.



are linearly independent, as $a_1 v_1 + a_2 v_2 \neq 0$ unless $a_1 = a_2 = 0$.



are linear dependent as $v_2 = v_1 + v_3 \Rightarrow v_1 + (-1)v_2 + v_3 = 0$ but $(1, -1, 1) \neq 0$.

Q5. Recall $\mathbb{C}$ is a v.s. over $\mathbb{R}$ or $\mathbb{C}$.

(i) Is $\{1+i, 1-i\}$ linearly independent over $\mathbb{R}$? Yes

(ii) Is $\{1+i, 1-i\}$ linearly independent over $\mathbb{C}$? No.

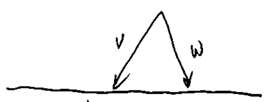
Q6. Let V be a vector space. Show that a subset $W \subseteq V$ is a vector subspace of V iff $\operatorname{span} W = W$.

pf. If W is a subspace of V, then every linear combination in W must stay in W. So $\operatorname{span} W \subseteq W$. Obviously $W \subseteq \operatorname{span} W$.

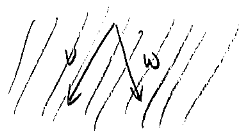
Conversely, if $W = \operatorname{span} W$, then for $u, w \in W$, $u + w \in \operatorname{span} W = W$; for $a \in \mathbb{F}$, $w \in W$, $a * w \in \operatorname{span} W = W$; $0 = 0_{\mathbb{F}} * w \in \operatorname{span} W = W$. Hence W is a subspace. $\square$

Rmk. convex combination — affine combination — linear combination

$\{(1-t)v + tw \mid t \in [0,1]\}$.



$\{(1-t)v + tw \mid t \in \mathbb{R}\}$
$\quad ||$
$\{av + bw \mid a + b = 1, a, b \in \mathbb{R}\}$ ← 1 degree of freedom



$\{av + bw \mid a, b \in \mathbb{R}\}$. ← 2-degree of freedom.

Tutorial 1.

Additional Materials.

This notes focus on examples and problems on top of the tutorial notes last year.

Q1. Verify this is a vector space over $\mathbb{F}=\mathbb{R}$.

$(V=\{f:\mathbb{R}^2\to\mathbb{R} \mid f \text{ is a continuous function}\}, \oplus, *)$ with

- $(f_1\oplus f_2)(\vec{x}) := f_1(\vec{x}) \overset{\text{addition in } \mathbb{R}}{+} f_2(\vec{x})$ for all $\vec{x}\in\mathbb{R}^2$ (definition of $f_1\oplus f_2$)
- $(a*f)(\vec{x}) := a \overset{\text{multiplication in } \mathbb{R}.}{\cdot} f(\vec{x})$ for all $\vec{x}\in\mathbb{R}$, $a\in\mathbb{R}$. (definition of $a*f$.)

pf. (Well-definedness). From analysis we see $f_1\oplus f_2$, $a*f\in V$ as they are both continuous function $\mathbb{R}^2\to\mathbb{R}$.

(VS1) $$[(f_1\oplus f_2)\oplus f_3](\vec{x}) \underset{\text{def of }\oplus.}{=} (f_1\oplus f_2)(\vec{x}) \underset{+\text{ in }\mathbb{R}.}{+} f_3(\vec{x}) \underset{\text{def of }\oplus}{=} (f_1(\vec{x}) \underset{+\text{ in }\mathbb{R}}{+} f_2(\vec{x})) + f_3(\vec{x})$$

$$\overset{\text{associativity of + in } \mathbb{R}}{=} f_1(\vec{x}) + (f_2(\vec{x})+f_3(\vec{x})) \overset{\text{def of }\oplus}{=} f_1(\vec{x}) + (f_2\oplus f_3)(\vec{x}) \overset{\text{def of }\oplus}{=} [f_1\oplus(f_2\oplus f_3)](\vec{x})$$

This is true for all $\vec{x}\in\mathbb{R}$, so $(f_1\oplus f_2)\oplus f_3 = f_1\oplus(f_2\oplus f_3)$, $f_1,f_2,f_3\in V$.

(VS2). $$(f_1\oplus f_2)(\vec{x}) \overset{\text{def of }\oplus}{=} f_1(\vec{x})+f_2(\vec{x}) \overset{\text{commutativity of + in } \mathbb{R}.}{=} f_2(\vec{x})+f_1(\vec{x}) \overset{\text{def of }\oplus}{=} (f_2\oplus f_1)(\vec{x}).$$

This is true for all $\vec{x}\in\mathbb{R}$, so $f_1\oplus f_2 = f_2\oplus f_1$, $f_1,f_2\in V$.

(VS3). Define $z:\mathbb{R}^2\to\mathbb{R}$, $\vec{y}\mapsto 0$, the zero map.

We want to show $z\oplus f = f$ for all $f\in V$.

$$(z\oplus f)(\vec{x}) \overset{\text{def of }\oplus}{=} z(\vec{x})+f(\vec{x}) \overset{\text{def of } z}{=} 0+f(\vec{x}) \overset{\text{identity of 0 in } \mathbb{R}.}{=} f(\vec{x})$$

As this is true for all $\vec{x}$ and all $f\in V$. We show that z is the additive identity for $\oplus$.

(VS4). For any $f\in V$, define $g:\mathbb{R}^2\to\mathbb{R}$, $\vec{y}\mapsto \overset{\text{inverse in } \mathbb{R}.}{-} f(\vec{y})$

We want to show

$$(f\oplus g) = z \quad (\text{and by (VS2), } g\oplus f = z)$$

$$(f\oplus g)(\vec{x}) \overset{\text{def of }\oplus}{=} f(\vec{x})+g(\vec{x}) \overset{\text{def of } g.}{=} f(\vec{x})+(-f(\vec{x})) \overset{\text{def of } - \text{ in } \mathbb{R}}{=} 0 \overset{\text{def of } z}{=} z(\vec{x}).$$

As this is true for all $\vec{x}$, so $f\oplus g = z$.

<u>Write $-f := g$. in sequel.</u>

(VS5). $(a*(b*f))(\vec{x}) \overset{\text{def of } *}{=} a\cdot\big((b*f)(\vec{x})\big) \overset{\text{def of } *}{=} a\cdot(b\cdot f(\vec{x})) \overset{\text{associativity of } \cdot \text{ in } \mathbb{R}}{=} (a\cdot b)\cdot f(\vec{x}) \overset{\text{def of } *}{=} ((a\cdot b)*f)(\vec{x})$.

product in $\mathbb{R}$.

Note that you cannot jump to $a * b\cdot f(\vec{x})$ here!
As $a \in \mathbb{R}$, $b\cdot f(\vec{x}) \in \mathbb{R}$
but $*$ is not an operation in $\mathbb{R}$.

This is true for all $\vec{x}$, so $a*(b*f) = (a\cdot b)*f$, $a, b \in \mathbb{R}$, $f \in V$.

(VS6). $(1*f)(\vec{x}) \overset{\text{def of } *}{=} 1\cdot f(\vec{x}) \overset{\text{identity of } 1 \text{ in } \mathbb{R}}{=} f(\vec{x})$.

This is true for all $\vec{x}$, so $1*f = f$.

(VS7). $(a*(f_1 \oplus f_2))(\vec{x}) \overset{\text{def of } *}{=} a \underset{\text{multiplication in } \mathbb{R}}{\cdot} (f_1 \oplus f_2)(\vec{x}) \overset{\text{def of } \oplus}{=} a\cdot\big(f_1(\vec{x}) + f_2(\vec{x})\big) \overset{\text{distribution laws in } \mathbb{R}}{=} a\cdot f_1(\vec{x}) + a\cdot f_2(\vec{x})$

$\overset{\text{def of } *}{=} (a*f_1)(\vec{x}) \underset{\text{addition in } \mathbb{R}}{+} (a*f_2)(\vec{x}) \underset{\text{def of } \oplus}{=} (a*f_1 \underset{\text{addition in } V}{\oplus} a*f_2)(\vec{x})$.

As this is true for all $\vec{x}$, we have $a*(f_1 \oplus f_2) = a*f_1 \oplus a*f_2$ for $a \in \mathbb{R}$, $f_1, f_2 \in V$.

(VS8) $((a \underset{\text{addition in } \mathbb{R}}{+} b)*f)(\vec{x}) \underset{\text{def of } *}{=} (a+b)\cdot f(\vec{x}) \underset{\text{distributivity of } + \text{ and } \cdot \text{ in } \mathbb{R}}{=} a\cdot f(\vec{x}) + b\cdot f(\vec{x}) \overset{\text{def of } *}{=} (a*f)(\vec{x}) \underset{\text{addition in } \mathbb{R}}{+} (b*f)(\vec{x}) \underset{\text{def of } \oplus}{=} (a*f \oplus b*f)(\vec{x})$

As this is true for all $\vec{x}$, we have $(a \underset{\text{addition in } \mathbb{R}}{+} b)*f = a*f \underset{\text{addition in } V}{\oplus} b*f$. $\square$

Remark. (i) $\mathfrak{z} \in V$ is called <u>zero vector</u> in V. Note that this is not the same as $0 \in \mathbb{R}$, as $0 \notin V$.

(ii) for any $f \in V$, $(-f)$ is called <u>additive inverse</u> to f in V with respect to the zero element $\mathfrak{z}$.

(iii). Two functions $f_1, f_2 \in V$ are defined to be <u>equal</u>

$$f_1 = f_2$$

if for all $\vec{x} \in \mathbb{R}^2$, $f_1(\vec{x}) = f_2(\vec{x})$.

This is why I added a sentence after every axiom I checked.

Q2. In the vector space in Q1. Verify $f_1, f_2, f_3 \in V$ are linearly independent over $\mathbb{R}$

where $f_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto \sqrt{x^2+y^2}$

$f_2 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto |x+y|$.

$f_3 : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto 1$. constant function.

Pf. f_1, f_2, f_3 are continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. so $f_1, f_2, f_3 \in V$.

For arbitrary $a_1, a_2, a_3 \in \mathbb{R}$.

the zero function $z : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto 0$.

Assume $a_1 * f_1 \oplus a_2 * f_2 \oplus a_3 * f_3 = z$. ①.

Then $(a_1 * f_1 \oplus a_2 * f_2 \oplus a_3 * f_3)(1,1) = a_1 \cdot f_1(1,1) + a_2 \cdot f_2(1,1) + a_3 f_3(1,1)$

$$= \sqrt{2}\, a_1 + 2a_2 + a_3 = 0$$

$\|$

$z(1,1) = 0$

so we have $\sqrt{2} a_1 + 2a_2 + a_3 = 0$. ②

Similarly evaluate both sides of ① on $(0,1)$ and $(0,0)$ gives

$a_1 + a_2 + a_3 = 0$. ③.

$0 + 0 + a_3 = 0$ ④

By ②-④, we see $a_1 = a_2 = a_3 = 0$.

As we start with arbitrary a_1, a_2, a_3, we have shown f_1, f_2, f_3 are linearly independent.

□.

Remark. It is very important that a_1, a_2, a_3 are arbitrary, and deduce that $a_1 = a_2 = a_3 = 0$.

Exercise: ①. $V_1 = \{ f : \mathbb{R}^2 \to \mathbb{R}$ smooth $| f'' + f' = 0 \}$ with $(f_1 \oplus f_2)(\vec{x}) := f_1(\vec{x}) + f_2(\vec{x})$.

$(a * f)(\vec{x}) = a \cdot f(\vec{x})$, $a \in \mathbb{R}$, $f_1, f_2 \in V$

②. $V_2 = \{ (a,b) \in \mathbb{R}^2 \mid a^2 - b^2 = 0 \}$ with $(a_1, b_1) \oplus (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

$a \cdot (a_1, b_1) = (aa_1, ab_1)$, $a \in \mathbb{R}$, $(a_1, b_1), (a_2, b_2) \in V$.

Show that $(V_1, \oplus, *)$ is a vector space over $\mathbb{R}$.

$(V_2, \oplus, *)$ is not a vector space over $\mathbb{R}$.

You also need to check $\oplus$ and $*$ is well-defined.